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TRANSLATION

THE SOLUTION OF BOUNDARY LAYER EQUATIONS BY THE DIFFERENCE METHOD

By

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FOREIGN TECHNOLOGY DIVISION

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THE SOLUTION OF BOUNDARY LAYER EQUATIONS BY THE DIFFERENCE METHOD

I. Yu. Brailovskaya and L. A. Chydov

In this article we present the methods and results of the numerical solution of the system of equations of a two-dimensional steady-state boundary layer on a plate. The examined method can be applied in the particular case of a homogeneous incompressible fluid and in the general case of a multi-component compressible gas taking into account heat transfer and diffusion.

The system of equations of the boundary layer in an incompressible fluid, as we know [1], has the form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2}$$

The boundary conditions, taking into account the possible passage of the fluid across the surface of the plate (so-called injection), are set as follows:

$$u=0,\quad v=v_w(x) \text{ when } \quad y=0, \tag{3}$$

$$u=U(x) \text{ when } \quad y=\infty \tag{4}$$

(here $v_w(x)$ is the transverse velocity of the fluid on the plate surface, differing from zero when injection is involved).

In the case of a compressible gas with heat transfer and diffusion, for a binary mixture, the boundary layer equations in dimensionless form (see [2]) are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}, \tag{5}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{6}$$

$$\rho u \frac{\partial c}{\partial x} + \rho v \frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(\rho D_{12} \frac{\partial c}{\partial y}\right) \frac{1}{\mathrm{Pr}_{g\infty}}, \tag{7}$$

$$\rho \bar{c}_p \left(u \frac{\partial \vartheta}{\partial x} + v \frac{\partial \vartheta}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda \frac{\partial \vartheta}{\partial y}\right) \frac{1}{\mathrm{Pr}_\infty} +$$

$$+ \rho D_{12}\left(\frac{c_{p_1\infty}}{c_{p_1\infty}} c_{p_1} - c_{p_2}\right) \frac{\partial c}{\partial y} \cdot \frac{\partial \vartheta}{\partial y} \cdot \frac{1}{\mathrm{Pr}_{g\infty}} + \mu \left(\frac{\partial u}{\partial y}\right)^2 \frac{1}{km}. \tag{8}$$

The boundary conditions

$$u = 0; \quad v = v_w; \quad \frac{\partial c}{\partial y} = -(1-c)\frac{\mathrm{Pr}_{g\infty}}{(D_{12})_w}; \quad \vartheta = 1 \text{ when } y = 0, \tag{9}$$

$$u = 1; \quad c = 0; \quad \vartheta = 0 \text{ when } y = \infty. \tag{10}$$

Here we use the designations adopted by Avduyevskiy and Obroskova [2]. The unknown functions are u, v, c and θ; the remaining values are known functions of v, c, and θ, or are constants.

We will examine Systems (1)-(2) and (5)-(8) with corresponding boundary conditions (3), (4), and (9), (10), assuming that for a certain $x = x_0$ all of the unknown functions are given.

For the numerical solution of the boundary layer equations in an incompressible fluid, the method of integral equations and different variants of the Galerkin method are very effective. The success of these methods was due to the comparative simplicity of System (1), (2) and the simple construction of the functions describing the boundary layer in an incompressible fluid. Attempts to use similar methods to solve systems of type (4)-(8), as a rule, lead to unwieldy non-standard

calculations which are inconvenient to carry out on high-speed computers.

In recent years several reports have been published concerning the use of difference methods for numerical calculation of the solution of boundary layer equations (see, e.g., [3], [4], [8], [9]). The application of difference schemes in such problems is complicated by the fact that the boundary layer equations are peculiar when $y = 0$, since when $y = 0$ the coefficients of the derivatives of x become zero.

As a result, simple evident schemes cannot be applied close to boundary $y = 0$ since here the conditions at Δx, associated with the requirement that the difference scheme be stable, are found to be unsatisfiable in practice. In an earlier article [3] boundary layer equations of the Mises form are used. The above-indicated peculiarity disappears, but another difficulty arises since the second derivative of the unknown function reverts to infinity when $y = 0$. This prevents the use of difference schemes with small values of y. Because of this, Mitchell [3] and Baxter [4] used special methods to calculate unknown functions in groups lying close to the straight line $y = 0$.

In view of the unwieldiness of these methods, it is not convenient to use them in more general cases when the number of equations increases, for example in the case of a multi-component gas. In this article an implicit difference scheme is used which permits calculation of the entire boundary layer in a simple uniform way, convenient for digital computers. Implicit difference schemes for boundary layer calculation are also used in other articles [8] and [9].

The system of boundary layer equations in any case consists of one first-order equation (continuity equation) and several second-order equations. Each of these equations is quasi-linear (linear with respect

to derivative) relative to its "own" corresponding function, i.e., that function which enters the equation as the first derivative with respect to x and the second derivative with respect to y. The equations are connected only through coefficients which depend on unknown functions and also on their first derivatives with respect to y (as, for example, in Eq. (8)).

Each of these second-order equations can be expressed in the following general form:

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(d\frac{\partial f}{\partial y}\right) + e. \tag{11}$$

The boundary conditions can also be expressed in general form:

$$\alpha\frac{\partial f}{\partial y} + \beta f = \gamma \quad \text{when} \quad y = 0, \tag{12}$$

$$f = c = \text{const} \quad \text{when} \quad y = \infty. \tag{13}$$

We will describe the difference scheme used by us. We will introduce on plane (x, y) a basic rectangular network comprised of points with coordinates

$$x = x_0 + n\Delta x; \quad y = m\Delta y \quad (m,\ n = 0,1,2,\ldots).$$

At these points, which we will call integral for conciseness, we will calculate the values of the functions of f satisfying equations of type (11). To reduce the notation we will introduce the designation

$$f_m^n = f(x_0 + n\Delta x, m\Delta y). \tag{14}$$

In addition to the basic network we also need an auxiliary network of points with coordinates

$$\left.\begin{aligned} &x = x_0 + n\Delta x, \quad y = (m + 1/2)\Delta y \\ \text{and}\quad &x = x_0 + (n + 1/2)\Delta x, \quad y = m\Delta y \end{aligned}\right\} (m, n = 0, 1, 2, \ldots).$$

At these "half-interger" points we will calculate the coefficients a, b, d, and e. We will use designations analogous to (14). A little

later we will describe a specific method of calculating the coefficients with respect to the value of the functions of f given in the groups of the basic network; for simplicity of explanation we will thus assume that the coefficients a, b, d, and e are known.

Differential Equation (11) is approximated with the aid of a two-layer implicit six-point scheme:

$$a_m^{n+1/2}\frac{f_m^{n+1}-f_m^n}{\Delta x}+b_m^{n+1/2}\frac{s\,(f_{m+1}^{n+1}-f_{m-1}^{n+1})+(1-s)\,(f_{m+1}^n-f_{m-1}^n)}{2\,\Delta y}=$$

$$=s\,\frac{d_{m+1/2}^{n+1}(f_{m+1}^{n+1}-f_m^{n+1})-d_{m-1/2}^{n+1}\left(f_m^{n+1}-f_{m-1}^{n+1}\right)}{(\Delta y)^2}+ \tag{15}$$

$$+(1-s)\,\frac{d_{m+1/2}^n(f_{m+1}^n-f_m^n)-d_{m-1/2}^n(f_m^n-f_{m-1}^n)}{(\Delta y)^2}+e_m^{n+1/2},$$

where $\frac{1}{2}\leq s\leq 1$.

The boundary conditions are approximated by the difference equations

$$\alpha\frac{f_1^{n+1}-f_0^{n+1}}{\Delta y}+\beta f_0^{n+1}=\gamma, \tag{16}$$

$$f_{M(n+1)}^{n+1}=c. \tag{17}$$

Here M(n + 1) is the number of the last group point for which unknown functions in the layer with the number n + 1 are calculated. Details on the selection of M(n + 1) will be described below.

The calculation will be conducted layerwise, consecutively converting from the layer with the number n to the layer with number n + 1. Let us assume all of the unknown values on the layer with the number n are known. Then, if the values of the coefficients a, b, d, and e introduced in Difference Equations (15) are known, Systems (15), (16), (17) determine the values of f in the layer with the number n + 1. This system will be determined by the usual step method [5], [6].

We will find step coefficients in the recursion ratio

$$f_m^n = A_m^n f_{m+1}^n + B_m^n \qquad (18)$$

by known formulas consecutively from m to m + 1. Coefficients A_o and B_o are determined with the aid of Boundary Conditions (16). The values of f_m^n of the unknown function are calculated by the inverse step with respect to Relation (18) with the aid of Boundary Condition (17).

The number of points M(n + 1) in the layer with number (n + 1) is determined from the condition that the difference $f_{M(n+1)}^{n+1}$ should have a sufficiently small absolute value. This condition provides a smooth transfer of f to its limiting value.

In practice, M(n + 1) is selected as follows. First we will assume M(n + 1) = M' = M(n), we will find (as has been described previously) $f_{M'}^{n+1}$ and $f_{M'-1}^{n+1}$ and we will verify the condition of smooth conjugation:

$$|f_M^{n+1} - f_{M-1}^{n+1}| \leq \varepsilon, \qquad (19)$$

where ε is a certain small positive number. If (19) is not fulfilled, we will assume M = M" = M(n) + 1 and will find $f_{M''}^{n+1}$ and $f_{M''-1}^{n+1}$. Thus, we will replace the missing values of all unknown functions in the layer with number n with their limiting values. If condition (19) again is not fulfilled, we will increase M to unity, and so forth, until (19) is satisfied.

After we obtain, in the layer with number (n + 1), the values for all functions of f satisfying equations of type (11) the transverse velocity v is found from the difference equation

$$\frac{1}{2\Delta x}[(\rho u)_m^{n+1} - (\rho u)_{m-1}^{n+1}] + \frac{1}{2\Delta x}[(\rho u)_{m+1}^{n+1} - (\rho u)_{m+1}^{n}] + \frac{1}{\Delta y}[(\rho v)_{m+1}^{n+1} - (\rho v)_m^{n+1}] = 0, \qquad (20)$$

which approximates the continuity equation (the density ρ entering into this formula depends only on the temperature and concentration of the

components, i.e., on functions of f, and therefore we now can consider ρ as a known value, as are the coefficients a, b, d, and e of the equations of type (11)). By Eq. (20) we can express $v_{m+1}^{n+1/2}$ in terms of $v_m^{n+1/2}$. The initial value of $v_0^{n+1/2}$ is known because of one of Boundary Conditions (3). Thus, we will find v in the layer with half-integer number n + 1/2 at the points with whole number m.

We will now explain the method of calculating the coefficients entering into Difference Equations (15). Obviously from Eqs. (5)-(8) the coefficients a and d depend only on the functions of f; the coefficients b and e can depend on derivatives of the type df/dy. For the calculation of the values

$$a_m^{n+1/2},\quad b_m^{n+1/2},\quad d_{m+1/2}^{n+1}\ \text{и}\ e_m^{n+1/2}$$

we must know the values of the functions of f and ρ in the groups of the auxiliary network. Using linear interpolation, we will assume:

$$f_m^{n+1/2} = \frac{1}{2}(f_m^n + f_m^{n+1}), \tag{21}$$

$$f_{m+1/2}^{n+1} = \frac{1}{2}(f_m^{n+1} + f_{m+1}^{n+1}). \tag{22}$$

For the calculation of $e_m^{n+1/2}$ we will assume:

$$\left(\frac{\partial f}{\partial y}\right)_m^{n+1/2} = \frac{1}{2}\,\frac{f_{m-1}^n - f_{m-1}^n}{\Delta y} + \frac{1}{2}\,\frac{f_{m+1}^{n+1} - f_{m-1}^{n+1}}{\Delta y}. \tag{23}$$

Here, naturally, a difficulty arises; for calculation of the functions of f in the layer (n+1) we must know the coefficients which themselves depend on unknown values of the functions of f. This difficulty is surmounted in the usual manner — with the aid of so-called enumerations or iterations.

In the first approximation, in the right-hand part of (21), (22) and (23) the superscript n + 1 is replaced by n and we find as previously described, the first approximation for f in the layer (n + 1), and v in the layer (n + 1/2). We will substitute the first approxima-

tion for $\underline{f}$ in place of the values of $\underline{f}$ in the layer $(n + 1)$ in Formulas (21), (22), and (23), we will find the second approximation for $\underline{f}$ in layer $(n + 1)$, etc. For the calculation of the coefficients $\underline{a}$ and $\underline{b}$, we must know $\underline{v}$ in addition to $\underline{f}$ in layer $(n + 1/2)$. For the calculation of the first approximation we will use the values of $\underline{v}$ in the next known layer, i.e., in the layer with the number $n - 1/2$; in subsequent approximations we can use the previous approximations for $\underline{v}$.

We compiled 3-6 enumerations. The function of $\underline{v}$ changed most noticeably in the enumerations.

In practical calculations we used a scheme which varied somewhat from the described. The basic difference was the application of a non-uniform network.

All of the functions which describe the boundary layer undergo (with fixed $\underline{x}$) the greatest changes with small values y; with an increase of $\underline{y}$, all derivatives decrease. Thus the step of Δy also increases with an increase of $\underline{y}$. In each layer there were m_1 points with step Δy_1, m_2 points with step $\Delta y_2 = 2\Delta y_1$, and m_3 points with step $\Delta y_3 = 2\Delta y_2$. At "junctions," i.e., at the points $m = m_1$ and $m = m_2$, first- and second-order derivatives were approximated with the aid of the values of the functions at the points with numbers $m - 2$, m, $m + 1$ (second-order approximations relative to Δy). We will not present here the obvious changes which must be made in Difference Equations (15), (20) and recursion formulas for the step coefficients A_m, B_m for points at which the step with respect to Δy changes. The use of the variable step with respect to $\underline{y}$ made it possible to considerably reduce the total number of calculation points. At "junctions" we observed no disruption of the smoothness of the functions to be calculated. The width of the boundary layer, generally speaking,

increases with an increase of $\underline{x}$.

In the course of the calculations we changed the "averaging parameter" $\underline{s}$, appearing in Difference Equation (15), taking into account the nature of the change of the functions of $\underline{f}$ with respect to $\underline{x}$.

As is obvious from Eqs. (1), (2) or (5)-(8), the transverse velocity $\underline{v}$, when $x = x_0$, is determined by giving values of all remaining functions when $x = x_0$. Thus, for example, from (1) and (2) we will have:

$$-u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \qquad (24)$$

This ratio represents a usual linear differential equation relative to $v(x_0, y)$. Boundary Conditions (3) determine the initial value of $v(x_0, 0)$. Numerical Solution of Eq. (24) is difficult because the coefficient when $\frac{\partial v}{\partial y}$ reverts to zero when $y = 0$.

The initial function $v(x_0, y)$ can be obtained with the aid of the iterative process shown on p. 7. This process will give us $v(x_0 + \frac{\Delta x}{2}, y)$; for sufficiently small Δx we can identify this function with $v(x_0, y)$.

When calculating the initial function $v(x_0, y)$ we noticed a great stability of the iterative process relative to perturbation of the zero approximation for $\underline{v}$. Convergence of the iterative process was not disrupted even for very large deviations of the zero approximation from the true function $v(x_0, y)$. The error decreased with the rate of a geometric progression having an index close to 1/2.

The stability of Difference Equation (15) was investigated in a case of constant coefficients $\underline{a}$, $\underline{b}$, and $\underline{d}$ with constant steps Δx and Δy of the network, and with uniform boundary conditions. The investigation was easily conducted by means of the Fourier transform [7]. Stability occurs with an arbitrary convergence of Δx and Δy to zero.

Judging from the results of practical calculations, the facts distinguishing the actual problem from this model case (boundary conditions, variable step with respect to $\underline{y}$, presence of non-linear relations between equations of the system) do not disturb the stability of the scheme.

The error of the method presented above was verified by comparison of results obtained using various steps of Δy and Δx, and by comparison with known solutions found by other methods.

For a comparison with the familiar solution of the Blasius equation by the described difference method, we solved System of Equations (1)-(4) with a zero pressure gradient and zero injection velocity v_w. Calculations were made from $x_0 = 0.01$; the solution of the Blasius equation [1] was used as the initial data. The initial layer contained 16 nodal points, not counting the points on the wall, wherein $m_1 = 10$, $m_2 = 4$, $m_3 = 2$; the values of the corresponding steps were $\Delta y_1 = 0.04$, $\Delta y_2 = 0.08$, $\Delta y_3 = 0.16$, and $\Delta x = 0.005$.

Five steps according to $\underline{x}$ were made: the obtained results of function $\underline{u}$ did not differ from the solution of the Blasisus equation by more than 0.25%.

System of Equations (5)-(10), where $\frac{\partial p}{\partial x} = 0$, was solved by the above-mentioned difference method for several variants.

As the initial data in all variants, we used the solution obtained in a previous article [2] for a particular case of boundary conditions that permitted a self-similar solution.

The calculations were carried out from $x_0 = 0.01$.

1. For the segment $0.01 \leq x \leq 0.0725$ calculation was conducted with self-similar boundary conditions [2] with a step of $\Delta x = 0.0025$ according to $\underline{x}$. In the initial layer 18 points were taken, not counting

the points on the wall, wherein $m_1 = 10$, $m_2 = 5$, $m_3 = 3$; the steps according to $\underline{y}$ were $\Delta y_1 = 0.05$; $\Delta y_2 = 0.1$, $\Delta y_3 = 0.2$.

With an increase of $\underline{x}$, the width of the layer increased in the manner described above; the conditions of a smooth conjugation on the upper boundary of the layer were verified only for the concentration function $\underline{c}$, since the remaining functions go to the asymptote when $y \to \infty$ more rapidly than $\underline{c}$.

The functions θ, $\underline{c}$, and $\underline{u}$, being a solution of System (5)-(10) in the self-similar case, depend only on a certain variable similitude $\xi = \xi(x, y)$. Consequently, the values θ, $\underline{c}$ and $\underline{u}$ obtained by the difference method with various values of $\underline{x}$ (in various layers with respect to $\underline{x}$) should agree during conversion to this variable, if our solution agrees with the exact solution of System of Differential Equations (5)-(10).

From the exactness of the coincidence we can judge the error of the method. An examination of the calculation results shows that on the segment $0.03 \leq x \leq 0.0725$ "scattering" of the values of functions for identical ξ does not exceed 2.5% for $\underline{u}$, 3% for $\underline{c}$, and 8% for θ, for the values of θ in the interval $1 > \theta > 0.5$ "scattering" does not exceed 3%. (The indicated values of θ belong to the interval $0 < y < 0.46$ where $x = 0.01$; the total thickness of the layer $\underline{x}$, in this case is equal to 1.6.)

The difference from the solution obtained in a previous article [2] did not exceed 3.5% for $\underline{u}$, 4% for $\underline{c}$, and 10% for θ. The change of the averaging parameter $\underline{s}$ did not noticeably influence the calculation results.

2. With the same distribution of points in the initial layer as in case 1, a calculation was conducted with Boundary Conditions (9),

(10) where we assumed v_w = const (these boundary conditions are not self-similar). The step according to $\underline{x}$ was taken as constant and equal to 0.0025. The calculation was conducted up to x = 0.1625. The width of the boundary layer was increased from L = 1.6, when $x = x_0 =$ = 0.01, to L = 7.2, when x = 0.165. The value of L in this and in all other variants was determined from the condition of smooth conjugation for the concentration function $\underline{c}$ of the upper boundary of the boundary layer.

The first six layers according to $\underline{x}$ were calculated with six iterations and s = 1 in order to acquire rapid attenuation of the initial error. Then we assumed that x = 0.07 and the calculations were carried out with three iterations. The calculation results are presented in the table.

3. The third variant differs from the second in that with a doubling of the boundary layer thickness in comparison with the original the steps according to $\underline{y}$ doubled. At the same time the step according to $\underline{x}$ doubled. This procedure was repeated for each recurrent doubling of the thickness of a boundary layer. The initial step according to $\underline{x}$ equaled 0.0025. In this variant the calculations were carried to x = 0.245.

A comparison of variants 2 and 3 showed that the difference in the values of $\underline{u}$ did not exceed 5% for all $\underline{x}$; the same was true for $\underline{v}$. The difference in the values of $\underline{c}$ for all $\underline{x}$ does not exceed 7% for values of this function larger than 0.1. The difference in the values of θ for all $\underline{x}$ does not exceed 7% for values of this function larger than 0.3.

For small values of $\underline{c}$ and θ pertaining to large values of $\underline{y}$, the difference in the results of these variants increases. The reason obviously is

that the step according to $\underline{y}$ far from the wall is too great in variant 3.

In a direction from the upper boundary of the layer to the wall the error rapidly diminishes as is obvious from a comparison of the results of these variants. We can judge the rate of diminishing according to the value of the step coefficients A_m, since, as is obvious from (18), the introduction of error ε into the values $\underline{c}$ and 0 when $m = m_0$ causes the appearance of an error of the order $\prod_{k=m_0-r}^{m_0} A_k \varepsilon$ in these values when $m = m_0 - r$.

In our calculations, in the examined region far from the wall A_m changed from 0.016 to 0.08 for the function θ and from 0.012 to 0.06 for the function $\underline{c}$.

For estimations of the influence of the averaging parameter $\underline{s}$ in the third variant after the sixth layer we assumed $\underline{s}$ equal to 1/2. The results obtained in this case (the calculation was carried to $x = 0.18$) differed little from the corresponding results of the third variant.

4. In order to verify the accuracy of choice of step Δy in the initial phase of the calculation, control calculation with a more dense network (according to $\underline{y}$) was carried out in the segment $0.01 \leq x \leq 0.02$.

In the initial layer 36 points were taken, with $m_1 = 20$; $m_2 = 10$; $m_3 = 6$; the steps according to $\underline{y}$ were taken equal to 0.025, 0.05, and 0.1, respectively. The step according to $\underline{x}$ was equal to 0.0025. The calculation was carried out with six iterations where $s = 1$. The results obtained in this case for the functions θ, $\underline{c}$, $\underline{u}$, and $\underline{v}$ differ very insignificantly from the results of variants 2 and 3 at the corresponding points of the network.

Below is the table of the calculation results for variant 2.

In the calculation we used the following formulas for determining the coefficients of the equations, depending on $\underline{u}$, $\underline{c}$, and θ:

$$\lambda = \frac{185c^3 + 37,1c^2 + 2,04c + 0,0305}{19,9c^3 + 8,0c^2 + 0,912c + 0,031} \cdot \frac{1,21}{1,2 - 0,5\theta}(1 - 0,5\theta)^{1.5}$$

$$\rho = \frac{1}{(1 - 0,5\theta)(1 + 13,5c)}.$$

$$\mu = \frac{1,2}{1,2 - 0,5\theta}(1 - 0,5\theta)^{1.5}\left(\frac{0,479}{1 + 0,13\frac{1-c}{c}} + \right.$$

$$\left. + \frac{1}{1 + 3,95\frac{c}{1-c}}\right)$$

$$\bar{c}_p = (1 - 0,5\theta)^{0.19}(13,15c + 1),$$

$$D_{12} = (1 - 0,5\theta)^{1,72},$$

$$\frac{c_{p_1\infty}}{c_{p_2\infty}} \cdot c_{p_1} - c_{p_2} = 13,15(1 - 0,5\theta)^{0.19}$$

$$k = 0,816326, \quad \mathrm{Pr}_{g\infty} = 0,151, \quad \mathrm{Pr}_{\infty} = 0,73.$$

The derived formulas pertain to the case of a hydrogen-air mixture (see [2]).

$x=0,0125$

y	θ	c	u	v
0,0	1,00000	0,41329	0,00000	4,2880
0,1	0,94315	0,39658	0,21135	4,0584
0,2	0,87034	0,20323	0,44293	5,3902
0,3	0,77103	0,13539	0,66056	8,1549
0,4	0,63143	0,05954	0,82542	11,375
0,5	0,46479	0,05980	0,92265	14,123
0,6	0,29527	0,01024	0,96970	15,903
0,7	0,16132	0,02722	0,98878	17,093
0,8	0,07685	0,01825	0,99634	18,052
0,9	0,03309	0,01207	0,99878	19,095
1,0	0,01569	0,00787	0,99939	20,131
1,2	0,00306	0,00312	0,99989	21,816
1,4	0,00058	0,00113	0,99998	22,854
1,6	0,00011	0,00036	0,99999	23,345
1,8	0,00000	0,00000	1,00000	23,445
		$x=0,0425$		
0,0	1,00000	0,62526	0,00000	4,2880
0,1	0,98164	0,54688	0,06373	3,9923
0,2	0,95982	0,46682	0,13925	3,9570
0,3	0,93401	0,39007	0,22565	4,2393
0,4	0,90365	0,32048	0,32077	4,8654
0,5	0,86818	0,26016	0,42133	5,8209
0,6	0,82699	0,20965	0,52287	7,0488
0,7	0,77944	0,16837	0,62039	8,4491
0,8	0,72492	0,13517	0,70392	9,9079
0,9	0,66308	0,10877	0,78438	11,314
1,0	0,59419	0,08790	0,84411	12,572
1,2	0,44257	0,05764	0,92438	14,442
1,4	0,29363	0,03806	0,96614	15,663
1,6	0,17181	0,02515	0,98582	16,482
1,8	0,08873	0,01650	0,99439	17,116
2,0	0,01084	0,01066	0,99789	17,666
2,2	0,01702	0,00673	0,99924	18,147
2,4	0,00653	0,00414	0,99971	18,545
2,6	0,00231	0,00244	0,99991	18,851
2,8	0,00078	0,00136	0,99997	19,072
3,0	0,00024	0,00069	0,99999	19,222
3,2	0,00005	0,00028	0,99999	19,319
3,4	0,00000	0,00000	1,00000	19,339
		$x=0,0725$		
0,0	1,00000	0,76164	0,00000	4,2880
0,1	0,99101	0,71081	0,02948	4,1001
0,2	0,98062	0,65309	0,06506	3,9771
0,3	0,96861	0,59292	0,10721	3,9444
0,4	0,95178	0,52978	0,15606	4,0273

$x=0{,}0725$

y	θ	c	u	v
0,5	0,93590	0,46695	0,21140	4,2474
0,6	0,92074	0,40067	0,27262	4,6231
0,7	0,90004	0,35061	0,33873	5,1536
0,8	0,87656	0,29588	0,40813	5,8314
0,9	0,85004	0,25498	0,48018	6,6382
1,0	0,82022	0,21595	0,55220	7,5164
1,2	0,74896	0,15392	0,68717	9,4795
1,4	0,66055	0,10954	0,79820	11,322
1,6	0,55544	0,07825	0,87907	12,854
1,8	0,43933	0,05620	0,93216	14,002
2,0	0,32295	0,04055	0,961[illegible]	14,808
2,2	0,21861	0,02931	0,[illegible]3	15,369
2,4	0,13352	0,02113	0,99143	15,788
2,6	0,07684	0,01514	0,99609	16,137
2,8	0,03925	0,01072	0,99829	16,453
3,0	0,01918	0,00747	0,99928	16,741
3,2	0,00857	0,00511	0,99971	16,997
3,4	0,00359	0,00340	0,99989	17,214
3,6	0,00142	0,00219	0,99996	17,391
3,8	0,00052	0,00135	0,99998	17,528
4,0	0,00018	0,00077	0,99999	17,529
4,2	0,00005	0,00038	0,99999	17,697
4,4	0,00000	0,00012	0,99999	17,757
4,6	0,00000	0,00000	1,00000	17,806
		$x=0{,}1025$		
0,0	1,00000	0,85698	0,00000	4,2880
0,1	0,99493	0,82618	0,01478	4,1835
0,2	0,98914	0,79015	0,03276	4,1004
0,3	0,98255	0,74895	0,05439	4,0184
0,4	0,97505	0,70302	0,08012	4,0389
0,5	0,96654	0,65315	0,11024	4,0852
0,6	0,95689	0,60019	0,14501	4,2028
0,7	0,94597	0,54641	0,18446	4,4021
0,8	0,93367	0,49231	0,22847	4,6930
0,9	0,91985	0,43955	0,27674	5,0823
1,0	0,90437	0,38929	0,32873	5,5719
1,2	0,86775	0,29988	0,44065	6,8526
1,4	0,82242	0,22711	0,55673	8,4198
1,6	0,76674	0,17013	0,66774	10,1164
1,8	0,69913	0,12751	0,76493	11,7634
2,0	0,61885	0,09548	0,84285	13,2149
2,2	0,52731	0,07174	0,90042	14,3921
2,4	0,42879	0,05412	0,93906	15,2842
2,6	0,33010	0,01095	0,96545	15,9294
2,8	0,23896	0,03103	0,98096	16,3902
3,0	0,16184	0,02349	0,98994	16,7313

$x=0,1025$

y	θ	c	u	v
3,2	0,10223	0,01772	0,99189	17,005
3,4	0,06021	0,01327	0,99750	17,214
3,6	0,03311	0,00985	0,99882	17,463
3,8	0,01706	0,00721	0,99946	17,666
4.0	0,00828	0,00519	0,99976	17,851
4,2	0,00380	0,00367	0,99990	18,012
4,4	0,00166	0,00253	0,99996	18,148
4,6	0,00068	0,00168	0,99998	18,259
4,8	0,00027	0,00106	0,99999	18,346
5.0	0,00009	0,00062	0,99999	18,413
5,2	0,00003	0,00028	0,99999	18,456
5,4	0,00000	0,00000	1,00000	18,442

$x=0,1325$

0.0	1,00000	0,92620	0,00000	4,2880
0,1	0,99716	0,91022	0,00682	4,2301
0,2	0,99304	0,89099	0,01514	4,2035
0,3	0,99032	0,86815	0,02524	4,1821
0,4	0,98622	0,84143	0,03743	4,1607
0,5	0,98160	0,81063	0,05202	4,2056
0,6	0,97610	0,77575	0,06931	4,2650
0,7	0,97055	0,73693	0,08961	4,3661
0,8	0,96397	0,69473	0,11317	4,5176
0,9	0,95660	0,64964	0,14017	4,7287
1,0	0,94835	0,60251	0,17072	5,0077
1,2	0,92888	0,50643	0,24238	5,8158
1,4	0,90484	0,41397	0,32683	6,9524
1,6	0,87541	0,33093	0,42093	8,3978
1,8	0,83966	0,26033	0,51995	10,082
2,0	0,79653	0,20272	0,61805	11,893
2,2	0,74480	0,15702	0,70912	13,698
2,4	0,68347	0,12144	0,78817	15,370
2,6	0,61223	0,09401	0,85238	16,818
2,8	0,53217	0,07296	0,90141	17,998
3.0	0,44615	0,05679	0,93675	18,913
3,2	0,35864	0,04432	0,96095	19,592
3,4	0,27499	0,03465	0,97677	20,085
3,6	0,20023	0,02708	0,98666	20,443
3,8	0,13793	0,02113	0,99259	20,715
4,0	0,08981	0,01642	0,99603	20,935
4,2	0,05523	0,01268	0,99794	21,126
4,4	0,03209	0,00972	0,99896	21,301
4,6	0,01766	0,00736	0,99949	21,461
4,8	0,00924	0,00551	0,99976	21,608
5.0	0,00461	0,00405	0,99989	21,739
5,2	0,00219	0,00292	0,99995	21,854
5,4	0,00100	0,00205	0,99997	21,951

x=0,1325

y	θ	c	u	v
5,6	0,00014	0,00138	0,99999	22,031
5,8	0,00018	0,00088	0,99999	22,095
6.0	0,00006	0,00050	0,99999	22,145
6,2	0,00002	0,00022	0,99999	22,185
6,4	0.00000	0.00000	1,00000	22,187

x=0,1625

0.0	1,00000	0,97444	0.00000	4,2880
0,1	0,99867	0,96868	0,00222	4,2750
0,2	0,99717	0,96208	0,00491	4,2691
0,3	0,99549	0,95381	0,00826	4,2721
0,4	0,99361	0,94379	0,01231	4,2859
0,5	0,99149	0,93177	0,01722	4,3134
0,6	0,98912	0,91744	0,02314	4,3580
0,7	0,98645	0,90051	0,03028	4,4233
0,8	0,98346	0,88072	0,03882	4,5139
0,9	0,98012	0,85782	0,04896	4,6352
1,0	0,97637	0,83166	0,06094	4,7932
1,2	0,96751	0,76942	0,09136	5,2151
1,4	0,95648	0,69511	0,13176	5,9834
1,6	0,94286	0,61210	0,18523	6,9715
1,8	0,92616	0,52554	0,21615	8,3009
2.0	0,90586	0,44096	0,31942	10,062
2,2	0,88141	0,36293	0,40090	12,143
2,4	0,85213	0,29432	0,48735	14,513
2,6	0,81729	0,23619	0,57475	17,068
2,8	0,77604	0,18829	0,65876	19,674
3.0	0,72751	0,14961	0,73544	22,197
3,2	0,67108	0,11875	0,80138	24,519
3,4	0,60675	0,09434	0,85664	26,555
3,6	0,53542	0,07507	0,89967	28,262
3,8	0,45915	0,05987	0,93202	29,634
4.0	0,38103	0,04784	0,95537	30,697
4,2	0,30481	0,03827	0,97158	31,195
4,4	0,23425	0,03063	0,98244	32,081
4,6	0,17248	0,02448	0,98946	32,509
4,8	0,12144	0,01953	0,99385	32,827
5.0	0,08167	0,01551	0,99651	33,073
5,2	0,05247	0,01225	0,99807	33,276
5,4	0,03223	0,00960	0,99896	33,452
5.6	0,01895	0,00746	0,99946	33,611
5,8	0,01068	0,00573	0,99973	33,756
6.0	0,00579	0,00434	0,99986	33,887
6,2	0,00302	0,00324	0,99993	34,004
6,4	0,00152	0,00236	0,99997	34,106
6,6	0,00074	0,00167	0,99998	34,194
6,8	0,00034	0,00113	0,99999	34,267
7.0	0,00015	0,00071	0,99999	34,328
7,2	0,00006	0,00038	0,99999	34,376
7,4	0,00002	0,00013	0,99999	34,414
7,6	0.00000	0,00000	1 ,00000	34,455

REFERENCES

1. H. Schlichting. Teoriya pogranichnogo sloya. IL , 1956.

2. V. S. Avduyevskiy and Ye. I. Obroskova. Issledovaniye laminarnogo pogranichnogo sloya na poristoy plastinke s uchetom teplo- i massoobmena. Izv. AN SSSR, ser. mekhan. No. 4, 1960.

3. A. Mitchell and G. Thomson. Finite Difference Methods of Solution of the Vor Mises Boundary Layer Equation with Special Reference to Conditions mear a Singularity. ZAMP, 9, No. 1, 1958

4. D. C. Baxter, Flugge-Lotz I. Compressible Laminar Boundary-Layer Behavior Studied by a Finite Difference Method. ZAMP; 9, No. 5/61, 1958.

5. I. S. Berezin and N. P. Zhidkov. Metody vychisleniy, Vol. 2, Fizmatgiz, 1959.

6. R. D. Rikhtmayer. Raznostnyye metody resheniya krayevykh zadach. I L, 1960.

7. V. S. Raben'kiy and A. F. Filippov. Ob ustoychivosti raznostnykh uravneniy. Gostekhizdat, 1956.

8. R. F. Kramer and H. M. Lieberstein. Numerical Solution of the Boundary-Layer Equations Without Similarity Assumptions. JAS, No. 8, 1959.

9. D. C. Leigh, The Laminar Boundary-Layer Equations. A Method of Solution by Means of an Automatic Computer. Proc. Camb. Phil Soc. 51, 320, 1955.

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